Why students fail calculus

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Abstract: Received doctrine that permeates the literature and practice of K-12 mathematics education, if implemented uncritically, will obstruct the development of skills that are critical for the study of calculus. Two false dichotomies are of principle concern :

- **Blind rote is always bad and leads to error, whereas thinking about what you are doing is always good and leads to success**
- **Solving problems by algebra is bad, whereas concretizing problems by the use of physical and visual models is good.**

Introduction

This paper is an attempt to enlist the assistance of K-12 educators in an effort to reduce the appallingly high failure rate of students in calculus courses locally and nationwide. Any attempt to do so must consider the critical role those educators play in helping students acquire tools for success in college mathematics.

A principal dichotomy that looms large in discussions of mathematics education is “blind rote,” typically identified with formal algebraic manipulation, versus “conceptual understanding,” evidenced by the ability to concretize symbolic processes by reference to everyday experience. Most educators believe that adherence to blind rote is the source of student’s weak mathematics performance, which can be ameliorated only by insistence on understanding. Herein that belief is challenged by reference to specific and significant examples that arise in the calculus curriculum. One conclusion is that the standard dogma, if accepted and implemented uncritically by K-12 mathematics educators and curriculum developers, poses a clear and present danger to the mathematical development of K-12 students who plan to study calculus in college.

The important role of calculus in the college curriculum and the American job market can be inferred from two independently compiled statistics. In Fall 2000, 1,156,000 June high school graduates began their freshman year at a four year college in the United States. At the same time, about 463,000 students in those institutions enrolled in first semester calculus. Comparing the two numbers, one might estimate that forty per cent of freshman took calculus. In fact, the actual percentage was considerably lower, since a significant portion of students were re-taking first semester calculus, having failed the course previously one or more times The author is unaware of any reliable estimate of the number of students in this category. There is widespread agreement, however, that far too many students flunk calculus and are thereby precluded from pursuing their career of choice.

In the United States and worldwide, one or more years of calculus are required of students who wish to pursue programs in science, mathematics, engineering, business, architecture, medicine, secondary math education, and many other fields. Many of these students will use calculus as an integral part of their professional activities. Nearly all will use the analytic and algebraic skills honed in calculus courses, and it is a lack of algebraic skills that poses the principal obstacle to students' success in calculus.

It is borne out by long experience that students with flawless algebra skills usually do well in calculus, while those with even moderate weaknesses in algebra perform poorly. In this respect, mathematics is unique. An essay with a few spelling and grammar mistakes is still intelligible. A couple of missed notes in the performance of a piano sonata don't make a difference. In mathematics, however, there is much less room for error. If a student makes an algebra mistake at the beginning of a problem, the remainder of the solution may be rendered completely irrelevant to the stated problem. The plight of a student doesn't notice that the algebra has gone astray is analogous to that of a deaf pianist who doesn't notice her hands slipping laterally a key or two. Cacophony ensues.

This essay is an attempt by a mathematician with long experience teaching all levels of calculus to engage K-12 educators constructively in an outcome-oriented discussion of the direction of current mathematics education reform. The author is no defender of the existing order. For decades, if not centuries, there clearly have been severe deficiencies in the mathematics preparation of entering students in the nation's colleges and universities. However, it is yet more apparent that present trends in mathematics education will exacerbate rather than alleviate those deficiencies.

The means for improving students' success rate in calculus are already at hand and are embodied in most important and useful principle of mathematics reform: the need for multiple approaches to problem solving. In the author's view, the current wave of curriculum reform has interpreted and implemented this doctrine far too narrowly. The need for multiple approaches applies in a far wider context than most mathematics educators imagine. Indeed, the examples elucidated herein demonstrate that in a curriculum designed to enable students to succeed in calculus, few patterns of mathematical thought or practice are either all good or all bad, nor can they be isolated from one another. This essay will argue, among other unorthodoxies, that

- Relying on blind rote is often good and is sometimes necessary.
- The attempt to "understand" underlying principles is sometimes irrelevant and counterproductive.
- Manipulatives must be used with discrimination, for reliance thereupon can foster habits of mind that damage students' mathematical development.
- Understanding and symbol manipulation skill are closely related.

1. Pedagogy and history

This paper is in part a commentary on visions and perspectives of today's mathematics educators. In large part, new curricula are based on a "constructivist" approach to childhood education, articulated in the 1989 Standards of the National Council of Teachers of Mathematics. A definitive account of the role of constructivism in mathematics education is the foundational treatise *Reconstructing Mathematics Education* [1993] , by Prof. Cathy Twomey Fosnot of the City College School of Education. That influential work describes sample topics from a constructivist curriculum and offers detailed accounts of the experiences of K-6 teachers who participated in an NSF sponsored teacher training program.

One teacher comes to "realize how useless most things are out of context. How nothing makes sense mathematically if there's no concrete picture (even in one's mind) of what it is we're talking about." Another opines: "there seems to be no middle road between understanding and applying formulas."

In the entire text, there is not a single instance of algebraic notation being met on its own terms. Every fraction, every number, and every operation on numbers is reduced to pictures. Every instance of "understanding" is visual (drawing diagrams) or physical (using number sticks and other manipulatives), to the virtual exclusion of mathematical notation. Symbols are viewed as obscurantist. "Understanding" and "constructing meaning" are synonymous with modeling. Teachers are inculcated with a belief that algebraic manipulation can and should be avoided. In fact, one teacher, who enters the program knowing how to solve word problems by applying algebraic methods, verbalizes the following epiphany:

"I can solve it with algebra, but whenever I go to algebra it is a cop-out. It means that I can't derive what the problem is about at its core, just based on logic and the information given."

Insofar as these teachers' statements characterize the pedagogical philosophy of K-12 mathematics educators, it is critical to examine their relevance and validity for students who are relying on K-12 curricula to prepare for the study of calculus. The historical development of calculus illuminates the present discussion. It is generally understood that Newton and Leibniz, working independently, developed the main ideas of calculus. However, much less well known is that Isaac Barrow, Newton's professor, has prior claim on many of the foundational concepts of the subject. However, Barrow's name is obscured from history books because he failed to develop mathematical notation that would facilitate working with those foundational ideas and applying them to the solution of real world problems. In short, Barrow knew the concepts, but he didn't have the right symbols and algebraic methods to work with.

The basic ideas of calculus may be summarized as follows.

- The derivative describes a rate of change of a quantity with respect to time, as well as the slope of the tangent line to a curve.
- Integration is a summation process that can be used to find areas and volumes.

- Integration and differentiation are inverse processes.

For the benefit of an audience with little formal mathematical background, informal and algebra-free descriptions can convey the importance of these concepts and, in a general way, how they are applicable to scientific inquiry. The lack of precision achieved thereby is perhaps analogous to that of Isaac Barrow's version of calculus, but is appropriate for introductory courses in the history of mathematics, or liberal arts courses once characterized as "math for poets."

Students of poetry contribute mightily to civilization but are not the object of the current study. Today's calculus students are exposed to and are expected to master the symbol-rich and algebra-based methodology of Newton and Leibniz. It is their version of calculus, achieved by the reduction of geometrical ideas to algebraic notation, that has permitted the study and solution of significant problems, ranging from the motion of planets to the study of magnetic and electric fields, to the configuration of electrons orbiting the atomic nucleus, to the sophisticated mathematics demanded by medical imaging technology. All of these applications are completely intractable without the use of algebraic notation and techniques. Indeed, Newton's *Principia Mathematica* opens with the seemingly megalomaniacal statement: "Herein I explain the system of the world." That his grandiose claim is not an exaggeration is due in large part to his successful use of notation and formal methods to represent and quantify the problems of mathematical physics.

This brief historical summary suggests that the above quoted teacher's statement "I can solve it with algebra, but whenever I go to algebra it is a cop-out" is one of the more wrongheaded mantras in the literature of K-12 mathematics education. This dogma is profoundly counterproductive to the success of high school graduates who are tackling first semester calculus and for whom algebra is the only method that can be used to solve the large majority of problems in textbooks and examinations.

Anti-algebra bias is even more explicit in the statement "there seems to be no middle road between understanding and applying formulas." This suggested Manichean distinction between understanding and formalism is based on ignorance, for formalism and reference to visual and physical models work hand in hand to permit the resolution of difficult questions. Specifically, algebraic techniques are the essential tool for moving from a general and intuitive discussion of a physical problem to a focused and quantitative understanding with predictive power.

All of mathematical physics can be viewed as the use of symbolism to lead a bewildered traveler through a dark wood of difficult concepts. And that is precisely how algebra is used in freshman calculus. Today's and tomorrow's college students need to walk into their first calculus lecture with well honed formal and symbolic skills. Insofar as K-12 curricula and educators de-emphasize algebraic techniques and denigrate algebraic formalism, they will deprive high school graduates of the opportunity to pursue mathematics-based careers.

2. Examples from the calculus classroom

Here are a few of the algebra errors that appear frequently on calculus exam papers.

- Canceling the x's in the fraction $\frac{x+2}{x+4}$
- Solving $x^2 = 4$ to obtain $x = 2$ thereby omitting the negative solution
- Rewriting $x^2 + 13^2$ as $(x+13)^2$
- Rewriting $\sqrt{x^2 + 2^2}$ as $x + 2$.
- Dividing both sides of $x^2 = x$ by x , concluding that $x = 1$, thereby omitting the solution $x = 0$.

Do students make these errors because they adhere to "blind rote?" Would they be less likely to make these errors if they "understood" more? If so, what sort of understanding is required by students if they are to avoid these and similar errors?

A critical observation is that the rational expression $\frac{x+2}{x+4}$, whether it appears in a lecture, text, or exam, is presented to the student and must be confronted as a combination of symbols. In this it is unlike a numerical fraction, which in some *but not all* contexts could be thought of as part of a unit, or as an action on a thing, or as a property of a physical object. The expression $\frac{x+2}{x+4}$ may indeed arise from representation of a real-life model. In all but exceptional circumstances, however, it is either impossible or counterproductive for the problem solver to keep track of the relationship between the model and its symbolic representation.

To work successfully with symbolic expressions, students must decide which transformations (usually, but not always, simplifications) of the expression are both legitimate and useful for obtaining the desired answer. The decision process requires both intelligence and judgment. What it does not require, or even permit, is reference to experience. Relying on symbols, to many educators and curriculum developers, implies abandonment of the understanding of the underlying physical model. This is simply not the case in the study of non-trivial mathematical and scientific problems. Instead, the methodology of modern science is to convert an intuitive understanding of a physical or geometric principle into a symbolic representation, in order both to facilitate a deeper understanding of that principle as well as to transform that intuition into a testable prediction.

In college mathematics lectures and exams, students need to manipulate fluently dozens, if not hundreds, of symbolic expression, and so they will have many opportunities to make common mistakes similar to the five listed above. Student errors on calculus exams can usually be classified as either careless or systematic. An example of a careless error is miscopying the fraction $\frac{2x}{x+2}$ as $\frac{2+x}{x+2}$, the sort of error that occurs only occasionally, typically when a student turns the page of his examination booklet.

In contrast, the five errors listed above are both frequent and systematic, in the sense that students who make them once make them more or less consistently. For example, it is not unusual for a student to ignore the negative solutions of equations such as $x^2 = 4$ and $y^2 = 1$ that may arise in the solution of different problems on the same exam, or to incorrectly cancel expressions such as $\frac{x+2}{x+4}$ several times throughout the course of a semester's examinations.

In an attempt to expose the reasons for such errors, it is not easy to assert that students illegally cancel $\frac{x+2}{x+4}$ because they don't understand mathematical concepts. They certainly need to know the underlying principle: cancellation works only for common factors of numerator and denominator. However, a precise statement of this principle requires at least half a page of algebraic definitions, which in fact students should know. In practice, however, competent students don't think about the cancellation rule when they are doing mathematics, for they have developed, through long experience, the ability to automatically distinguish situations in which cancellation is legal from those in which it is not.

Students who hope to succeed in calculus need to develop an automatic facility for distinguishing not only between legal and illegal symbolic manipulations, but also for deciding which of several possible legal manipulations is appropriate to the problem of current interest. If they don't, they're going to have trouble getting through a calculus exam or homework assignment. Subsequent sections will address methods for helping students to acquire the tools necessary for achieving these two goals.

3. Blind rote

Standard dogma asserts that understanding is always the cure for mistakes that would otherwise arise from adherence to blind rote. The clearest counterexample involves substituting an expression for the argument of a function. This requires a bit of explanation.

A function can be thought of as a little black box that acts on a number to produce another number. For example, after carefully observing the path of a falling object, one could in principle express its height above the ground as a function of time. Deducing the precise form of the position function is the principle goal of celestial mechanics, ballistics, and all applications that require predicting from a priori information the position of a moving object at a given future time.

As an illustrative example, consider the x^2 button on most calculators, a button whose sole purpose in life is to square a number, i.e. to multiply that number by itself. Fulfilling this duty entails three steps:

1.The user types a number, called the **input,** which then appears in the display
2. The user presses the x^2 button.
3. The calculator display switches to the **output**, or **value**, of the function, which in this example is calculated by multiplying the input by itself.

In the absence of magic boxes such as calculators and computers, a first attempt to communicate to a human the action of the x^2 button might be:

"The value of the squaring function at a number is the number times itself."

Unfortunately, this language is much too cumbersome to be useful in practice. A more concise statement would be

Square of a number = number*number.

The word "of" is superfluous, and is indicated more clearly by parentheses:
Square (number) = number*number.

In mathematics, brevity is essential, both because it saves paper and because it allows the eye to scan more easily a complex expression. Therefore the function definition is abbreviated to something like:
S(n) = n*n, in which the words "square" and "number" have each been abbreviated by their initial letter.

In the algebra sentence S(n) = n*n, the first symbol is the name of the function. The name of the input is enclosed in parentheses. The output, the expression to the right of the equal sign, is an algebraic expression that can be figured out by applying arithmetic operations to the input. The entire sentence is the definition of the function S.

In practice, mathematical custom of long standing prefers using letters at the end of the alphabet to name a function input, and the most common usage in elementary texts is to write the squaring function as S(x) = x. Reader take note: contrary to certain usages in mathematics and literature, the letter 'x' does not indicate an unknown, but is rather a variable quantity that can assume any desired numerical value. For example, a student wishing to find the square of 5 might write

Step 1) S(x) = x * x, the function definition, and then
Step 2) S(5) = 5*5 to indicate that when x is 5, the output of the squaring function is 5*5, and finally

Step 3) $S(5) = 25$ to express the output as a number.

The alert reader will observe that the equals signs in 1) and 2) mean different things. Step 1) defines a function (a rule, if you will) whereas Step 2) asserts that the result of applying the function to the input 5 is the output 5*5. The distinction between these two usages is a possible source of confusion that should be addressed, in my view, by using alternate notation. In fact, some computer languages use the notation $S(x) := x*x$ to make it easier for the compiler (the program that translates from human language (including algebra) to machine language) to understand the programmer's intent to define a function. The remainder of this paper will employ that notation which, unfortunately, has yet to penetrate classrooms, texts, or journals of mathematics and science.

Step 2) , the principal concern of the present discussion, is called **argument substitution** This procedure is used thousands of times in any calculus text and perhaps a dozen times during a typical calculus examination, and so it deserves a precise description. The above example suggests the following description: replace each symbol 'x' in the function definition by the symbol for the input. Unfortunately, this fist attempt isn't quite correct. To see why, assume that the addition fact $5 + 4 = 9$ is temporarily unavailable, and try to evaluate (i.e., find the value of) the squaring function when its input is 5 + 4. According to the proffered description, the value of the squaring function is obtained as follows:

Step 1) $S(x) := x * x$ (the function definition)
Step 2) $S(5 + 4) = 5 + 4 * 5 + 4$ (Erase and replace each 'x' by '5 + 4.')

Unfortunately, the universal convention that multiplication is performed before addition forces the right hand side of Step 2) to be interpreted as $5 + 20 + 4$, which is 29, rather than as 9*9, or 81, the expected function value. To convey the desired order of operations, add before multiplying, it is necessary to use parentheses:
$S(5 + 4) = (5 + 4) * (5 + 4)$, after which the function value may be simplified to $9 * 9 = 81$.

Therefore a complete description of argument substitution is as follows.

Suppose a function is defined by S(x) = [expression involving x]
To find the value of the function at an input, erase each symbol 'x' to the right of the = sign and replace it by the input *enclosed in parentheses.*

A critical application of function notation is its use when the input is itself represented by an algebra expression rather than by a number, as was the case in the previous examples.

Example 2: Given $S(x): = x*x$, find $S(a + b)$ without simplifying.
Solution: $S(a + b) = (a + b) * (a + b)$.

Most students do this problem correctly. However, one occasionally encounters the following error:

Example 3: Given S(x) := x* x, find S(a+2)
Error: S(a+2) = a^2 + 2.

Finally, the following error is encountered with astonishing frequency in precalculus and calculus courses..

Example 4: Given $F(x) = (x+h) + \frac{1}{(x+h)}$, find f (x + h).

This example, typically the initial step of a much longer problem, involves a slight twist, for the proposed input x + h includes the same symbol x that denotes the argument in the function definition. Nevertheless, the argument substitution rule explains exactly what to do. Erase and replace each 'x' by '(x + h)' as follows.

$$F(x) = (x+h) + \frac{1}{(x+h)}$$

If a student faithfully follows the argument substitution rule, Example 4 is a giveaway. Therefore, it is surprising that many college students frequently go astray in this problem. In particular, the following error is encountered with astonishing frequency:

$$F(x) = x + \frac{1}{x} + h$$

In the typical context of this problem, the error cited transforms a long problem into a trivial one. There is no sensible way to assign partial credit. The student has made a systematic error that will result, if not corrected, in a downward spiral of that students' calculus grade and career aspirations. .

Why is argument substitution so difficult? In a general sense, students may be troubled by the failure of that symbolic process to reflect any model that occurs in nature. For example, the process of evaluating the function F(x) = x^3 + x^2 -3 when x = 2 cannot be illustrated by pictures or manipulatives. Indeed, models are irrelevant, for argument substitution is a purely formal process. In this writer's opinion and experience, it is pedagogically productive to describe argument substitution as a game played with symbols. It is not a frivolous game by any means. To the contrary, it is the single most important formal tool of mathematics and science. But it is a game that should be played by following the argument substitution rule to the letter.

Students probably are trapped by their attempt to make sense of a purely formal rule. In the incorrect solution above, the student writes F (x + h) and notices that 'h' is being added to something. Because the function definition looks like an equation, and because changing an equation requires doing the same thing to both sides, the student decides to treat the right side fairly treatment by adding h to it. Unfortunately for such a student, a function definition isn't an equation. There is absolutely nothing algebraic going on. Argument substitution must be carried out completely mechanically as a pure "erase and replace" operation that has nothing whatsoever to do with addition, subtraction, fractions, or any mathematical, arithmetical, or algebraic idea. It is simply a procedure, a recipe if you will, for altering the appearance of the original function definition.

My conclusion, shocking as it may sound to some, is that argument substitution should be carried out by "blind rote," and that attempting to understand what's going on may lead to errors. Indeed, the following standard dichotomy:

- Rote is bad, and leads to error.
- Thinking about what you are doing is good, and leads to success based on understanding.

is incorrect, or is at best irrelevant, for students who are attempting to master argument substitution. Indeed, a student who feels that she must "understand" every new mathematical idea is likely to fall into the sort of trap illustrated by the above example, whereas one accustomed to plugging in symbols and playing with them by following rules is more likely to carry out the argument substitution rule with ease and consistent accuracy.

Many subjects in mathematics offer ample opportunity to think deeply about mathematics and to proceed with eyes wide open. Argument substitution is most assuredly not one of them.

A somewhat similar example is taken from a calculus lesson taught about two years ago. As part of a standard discussion of why the absolute value function f (x) = |x| doesn't have a derivative at x = 0, the instructor wrote the following expression on the board

$\lim\limits_{h \to 0} \dfrac{|0+h|-|0|}{h}$ and then asked the class to help simplify the expression. The reader who doesn't understand this expression or the previous sentence is advised not to worry. What is important for participating as a student in the following dialogue is to pay careful attention to the instructor's language.

Instructor: Who can help me simplify this expression?
Students:(Silence.)

Instructor: (2nd try): Well, just do one step: anything that will make the expression simpler or shorter:
Students: (Silence.)

Instructor: (desperate, 3rd try): I don't care if you understand what you are doing. Inside that scary-looking formula is something that you see every day, a few symbols that can be replaced by just one symbol. Please!

Students: (many hands raised): Change 0 + h to h !

In my view, the students were catatonic because they were worried about the "meaning" of the expression, and so succumbed to a grave syndrome that I call "symbol shock." They were worried about what the limit symbol meant, or perhaps by the use of vertical bars to denote absolute value. The teacher's third attempt encouraged them to overcome their fears and realize that '0 + h,' even when embedded in a complicated expression, should be replaced by 'h.'[1]

At this juncture, the reader of a certain persuasion may well be thinking that the examples just considered are not the essence of mathematics; that that they involve useless playing with symbols; that a computer could do this sort of thing, and so forth. These concerns will be addressed in the following sections.

4. What is calculus about, and why?

In today's sometimes acrimonious discussions of mathematics education, one encounters assertions that the very nature of mathematics is itself a subject for debate. Some have gone so far as to misinterpret Wittgenstein to buttress a claim that mathematics is socially constructed.[][] For the purposes of this essay, all such philosophical discussion is irrelevant; rather, the concern is to prepare K-12 students for the specific conception of mathematics defined by the calculus curriculum at nearly all American colleges.

The referenced conception of calculus is not completely universal, for the de-emphasis of symbolic manipulation and algebraic techniques in 1989 NCTM Standards-based curricula was accompanied by a parallel attempt to implement a reform vision of calculus. The stated goal of Harvard Calculus Consortium to was to "blah blah a vision of calculus [in which understanding is emphasized rather than rote manipulation].' [] That curriculum has achieved extremely limited endorsement at American universities, for repeated experiences in many mathematics departments,including City College's, that have experimented with the Harvard curriculum indicate clearly that it fails to prepare students adequately for subsequent science and engineering courses.

A critical subtext of mathematics reform in the twenty-first century is that symbolic manipulation may once have been important, but is now in the process of being rendered obsolete by the advent of technology. Initially, the availability of calculators suggested to some that there is no longer a need for K-6 students to memorize, practice, or even to learn the traditional algorithms of arithmetic. Later, when computer algebra systems were implemented on hand calculators, the argument was extended to call into question the

[1] A more precise statement is that the symbolic expression "0+h" should be replaced by "h" whenever that expression signifies adding 0 to h. The reader is invited to devise an example in which this is not the case and in which the indicated replacement would be erroneous. The answer to this riddle is provided as an endnote.

need for students to learn algorithms for manipulating algebraic expressions and solving equations.

In this writer's view, the availability of computer technology is more of a curse than a blessing for the vast majority of college students. The buck stops in the nation's science and engineering classrooms. Whether the subject is chemistry, biology, physics, engineering, economics, or finance, a students' most challenging task is to follow and absorb page after page and blackboard after blackboard of formulas and equations. Science books cannot be de-algebraized for the simple reason that all technological innovation depends on an algebraic description of physical phenomena.

The K-12 traditional curriculum is structured, partly by design and partly by historical accident, to bring students to the level needed to achieve the facility with formal skills needed to speak effectively the language of science. It follows that the availability of computing devices that do either arithmetic or algebra is irrelevant to students' needs to be able to read and to listen to mathematical formalism. Although computer algebra systems have a legitimate place in advanced courses, they pose a clear liability to students who are not already skilled with algebraic manipulation. Students who don't get adequate practice doing (i.e., writing) algebra will encounter the greatest difficulty understanding texts and lectures, in much the same way that over- and early use of calculators correlates with and is a presumed cause of young students' failure to acquire basic arithmetic skills.[][]

Most disturbing to many university mathematicians is the suggestion, implemented by a number of NCTM Standards-based curricula, that traditional algorithms of arithmetic be de-emphasized or dropped entirely. The oft-stated objection that students should not waste time finding the answer to a problem that could be solved with a calculator entirely misses the core issue, which is the continuum of logic and skills development that runs through the curriculum, from arithmetic to algebra to calculus. If students don't learn the long division algorithm for whole numbers, they won't be able to understand the long division algorithm for polynomials when they reach that topic in high school algebra. If they don't know long division of polynomials, they won't have any feel for techniques of integration in elementary calculus or for Laplace transforms that are introduced in more advanced courses. Even if hand computation of the answer to a division problem is no longer important, the algebraic patterns and processes that lead to the answer are very important indeed.

In summary, K-12 students headed for calculus need to practice algebra because all advanced science texts speak a universal language that is written with variables and is punctuated with subscripts, superscripts, summation signs, and lots of complicated notation. If they're not comfortable with writing algebra, they won't be comfortable reading it. If they haven't devoted significant effort to hands-on manipulation of rational expressions, they will be in a total fog by the end of the first chapter of a typical science or engineering textbook.

Nothing stated in this essay should be taken to suggest that proficiency with the language of science is sufficient for success in science. Rather, such proficiency is simply an absolute necessity without which students cannot survive. While there is substantial room for improvement at all levels of the curriculum, there is no room whatsoever for a perspective that calculus students can make do without algebra, or even with less algebra than they currently encounter. Indeed, the accelerating mathematization of formerly descriptive sciences such as biology makes it all the more crucial for a growing cadre of students to acquire fluent algebra skills before they begin calculus.

Jelly beans and limits

In "Reconstructing Mathematics Education," [] Fosnot and Schifter discuss at length the classroom experience of Ginny, a teacher who models division by distributing jelly beans among class members. Such a model is the sensible sort of initial learning experience that any capable educator or parent would provide when introducing this subject. From the perspective of the current paper, the critical challenge of early mathematics education is to follow up these initial experiences with a gradual and appropriate transition to symbolic representations. To do so, it is critical to address the details of that transition process. The authors purport to do so in the following passage. .

"This is not to say that all mathematical activity must retain an immediate relationship to the physical world. After enough experience distributing objects, the children in Ginny's class will be able to imagine sharing without having to enact it, and reflection on their actions will allow them to generalize from specific cases to more abstract notions of division, although reference to their jelly bean problem can be called up as needed."

The abstract notions that Fosnot and Schifter refer to are not at all clear. Indeed, the present writer the present has the greatest difficulty with their inspiring yet totally unsubstantiated conclusion:

"In just a few years, children's understanding of division of whole numbers will form the basis for constructing an understanding of division of fractions, and later still, for algebraic rational expressions, hyperbolic functions, or the limit of 1/x as x increases without limit."

It is important to examine the credibility of the multiple transitions suggested in this visionary sentence. The first clause is problematic, for the transition from whole numbers to fractions requires a delicate balance of concrete representation and abstract notation. It is critical for children to understand the core notion that unit fractions such as 1/5 represent the result of equal subdivision of a unit into pieces, but it is also essential for them to encounter and absorb, as early as possible, the purely algebraic statement that 3/5 is an abbreviation for 3 * 1/5. To see why, it is useful to study a standard example of what is often attacked as blind rote: the rule for multiplying fractions.

If teacher training programs and student texts were true to the doctrine of multiple representation, they would observe that the rule for multiplying 3/5 * 4/7 is an immediate consequence of the notational conventional stated above. Indeed, that product is simply 3 * 1/5 * 4 * 1/7, which equals 3*4*1/5*1/7, and so the only remaining question is to devise a rule for determining the product 1/5 * 1/7. By means of a simple observation about notation, the original product of general fractions has already been reduced to a much easier case, namely the product of unit fraction, easily determined by experience. In the present example, if a unit stick is equally subdivided into 7 pieces, and each piece is equally subdivided into 5 pieces, the resulting pieces are the result of equal subdivision of the unit stick into 5*7 pieces, each of length 1/(5*7). Thus 1/5 * 1/7 = 1/(5*7). It follows immediately that the desired product is 3*4 * (1/(5*7)), which for the same reason as before is expressed in more compact fraction notation as 3*4/(5*7).

The argument just presented adheres to the principle of using multiple representations by combining a symbolic argument with one based on modeling the number line as a stick. The author would be grateful for a reference to this argument in existing curricula or methods books. On the contrary, mathematics educators have developed an uncritical adherence to a purely geometrical model: the cake model for multiplying fractions. From observations of teacher training sessions, this model is quite difficult to absorb, even for prospective K-6 teachers, for at least two reasons. First, the pictorial representation for multiplying general fractions is not consistent with that of multiplying proper fractions, in that the former produces a "product cake" that completely covers and therefore hides the original unit cake. Furthermore, the cake arguments unnecessarily implicates an area interpretation of fractions in a situation where a linear representation, as the length of a stick, is more than sufficient. Area is a simple and elegant concept if presented carefully[2] but leads to muddled thinking otherwise. Indeed, teachers in the observed training session failed to respond to the trainer's inquiry whether the fraction multiplication rule is based on counting or upon area.

While it is possible to entertain Fosnot and Schifter's suggestion that division of fractions can be modeled adequately by experience of sharing, the present author considers the remaining transitions, from fractions to rational expressions to the definition of limit, to be totally implausible. The transition from fractions to rational expressions is overwhelming, except in those few curricula that make an effort to give fourth or fifth graders a reasonable glimpse of algebraic notation. When a student first encounters a rational expression such as $\frac{x+8}{x+4}$ he enters a new world of notation in which it is impossible and misleading to suggest that the new notation refers to an activity of sharing. On the contrary, it is assuredly counterproductive to describe this expression as standing for the distribution of x + 8 jellybeans among x + 4 children. The most probable consequence of such an exposition is that students will block out the variables and focus on the numerals. Precisely for this reason, it is critical to impress upon students that the

[2] Area is a function that assigns to every piece of paper in the world a numerical value. In particular, there is a reference paper, a square whose area is 1. The area function is defined operationally as follows. Take any piece of paper and cut it into two pieces. Then the area of the original is the sum of the areas of the pieces. Curricula should be evaluated according to whether they articulate this or an equivalent definition.

expression $\frac{x+8}{x+4}$ is NOT related to a situation in which 8 is divided by 4. Indeed, it may well be an attempt to identify this expression with a concrete model that leads students to try rewriting the fraction as $\frac{x+2}{x+1}$. Only a careful exposition of factoring algebraic expressions will empower students to cancel only when legal and necessary.

The final transition, first from rational expressions to an understanding of hyperbolic functions, and second to understanding the limit of 1/x as x increases without limit, is nothing if not dramatic. The reference to hyperbolic functions is puzzling, for they occupy a minute corner of the calculus curriculum. It is the claim about understanding the definition of limit that demands close attention.

In recent years, most college mathematics instructors have encountered more and more students who are less and less well equipped to understand the rigorous definition of limit. Indeed, that definition has been banished from the standard first-semester calculus course at all but a handful of top-rated institutions, precisely because fewer and fewer students graduate high school with the algebraic skills needed for a rigorous formulation of that definition. The situation has degenerated to the point that a recent paper in JRME [] surveys student perceptions of the intuitive meaning of limits without eliciting a single response that correctly transcribes the precise definition of limit into everyday language. Instead, the best intuitive statement is:

The limit as x approaches 2 of f(x) = x*x is 4 because as x gets close to 2, then x*x gets close to 4,

Such a quasi-definition is a pale approximation to the truth that is dangerously imprecise and contains significant potential for misunderstanding, for it suggests that the inter-relationship of the two phrases

- x gets close to 2
- x*x gets close to 4

is that an observer first contemplates the distance from x to 2 and afterward verifies that x*x is close to 4. Precisely the opposite is the case: a hypothetical challenger offers an arbitrarily stringent criterion for x*x being close to 4, and the respondent must respond with a notion of x being close to 2 that ensures fulfillment of the challenger's demand. This correct formulation, to be tested and understood, must be translated to symbols to clarify methods for responding to the challenger's demands.

Unfortunately, an increasing portion of calculus instructors, confronted by students who immediately collapse into symbol shock when they see the variables ε and δ that are traditionally employed to represent the two distances in the definition of limit, are forced to revert to the mathematically imprecise quasi-definition. The only way to help students avoid paralysis is to provide them with a rich experience of symbol manipulation before they get to college. To the extent that reform curricula reduce the emphasis on algebra and notation, they will enlarge rather than reduce the cohort of students who are properly prepared for calculus. It is the responsibility of curriculum developers to provide a clear map for the transition from jellybeans to symbols.

What is to be done?

Given the increasing need for many more high school graduates to be completely fluent with algebra, how might one improve the algebra instruction that is currently achieving the desired impact on all too few students in American high schools? A feasible solution has been formulated properly by mathematics educators but to date has been interpreted far too narrowly: understanding is a critical component of mathematics education. The kind of understanding that students need to be expert in algebra is not obtained by reference to models. What is lacking is a coherent overview of algebra rather than the current practice of presenting algebra as a disjointed collection of techniques.

The suggestion to be proposed can best be understookd by working though a short algebra quiz. The reader is assured that the result will not become part of his final grade.

Every student who walks into calculus class should be able to solve all of the following equations in five minutes.

a) $3x = 5$

b) $3x + 5 = 0$

c) $x^2 = x$

d) $x^3 = x$

e) $x^2 + 4x = 5$

f) $x^2 + 4x = 6$

In fact, relatively few students complete this quiz without falling into one or more of a number of systematic errors.

a) is easy: divide both sides by 3 to get x = 5/3.

b) is drummed into students so often that they tend to get it correct as well, by subtracting 5 from both sides and then dividing by 3.

c) trips up quite a few students. By analogy with a), students divide both sides by x and conclude that x = 1. That's clearly wrong, since x could also be 0. Any approach to polynomial solving must constrain students to avoid this error. The error arises because fractions whose denominators are 0 are meaningless, undefined, or illegal, depending on mathematical context as well as linguistic taste.

d) is even worse. Again, students divide by x and miss the solution x = 0. Furthermore, the resulting equation, x^2 = 1, tricks unwary students into writing the solution as x = 1, whereas of course x = -1 is also a solution. Very few students find all three roots.

e) Students sometimes factor the left side to obtain x (x+4) = 5. This information is useless. Nevertheless, it is common to see such students proceed further by writing x = 5 and x + 4 = 5, or perhaps x = 5 and x + 4 = 1. These students know that factoring helps solve a quadratic equation. Unfortunately, the only equations that are ready to solve by factoring are equations in which one side is zero. The student who makes this mistake is indeed showing a lack of understanding, for he is generalizing the crucial principle:

If a product is zero, one of the factors is zero

to an incorrect but linguistically analogous statement about nonzero products. A correct solution is

$$
\begin{aligned}
&\text{e) } x^2 + 4x = 5\\
&x^2 + 4x - 5 = 0\\
&(x+5)(x-1) = 0\\
&x + 5 = 0 \text{ or } x - 1 = 0\\
&x = -5 \text{ or } x = 1
\end{aligned}
$$

In f), even students who correctly transform the problem to $x^2 + 4x - 6 = 0$ tend to get stuck because they try too hard to factor this expression. They have not learned students that solving a quadratic equation involves a decision process, the conclusion of which is to use the quadratic formula rather than try to factor.

Some students have an intuitive affinity for symbolic notation and solve the listed problems automatically, even with a minimum of formal instruction. The majority of students do not fall into this category. This group needs the careful attention of curriculum developers and mathematics educators.

In New York State, a first step would be to reconnect those parts of the algebra curriculum that have been torn asunder by Sequential Mathematics and its successor Maths A and B. The second and more important step would be to adopt a new perspective on what is needed to “understand” symbolic mathematics. To do so, students should be encouraged to demonstrate their understanding of algebra by writing exercises or classroom discussions in which they formulate explicitly both the definitions of algebraic concepts and a critical repertoire of decision processes. Furthermore, as soon as they learn about a topic, they should be required to write about common pitfalls and to explain how to avoid them. The Appendix suggests sample examinations and solutions for this type of curriculum.

The principal goal of this paper has been to expose as both misguided and counterproductive the rhetoric that identifies symbol manipulation with lack of understanding. Indeed, like any other subject, symbol manipulation can be studied and tested with or without understanding, and students are in dire need of a shift from the latter to the former paradigm.

From a pragmatic perspective, a reasonable albeit imperfect measure of students' understanding of algebra is their ability to perform symbol manipulation flawlessly. While one occasionally encounters an *idiot savant* in algebra, it is the author's firm but unsubstantiated belief that students who consistently perform algebra without errors are *prima facie* demonstrating an internalized understanding of definitions and decision procedures. Even for these students, it would be beneficial to articulate their understanding as suggested above.

For the far larger cohort of students whose algebra skills are not automatic, curriculum innovation that encourages students to verbalize concepts and decision procedures, avoids a disjointed presentation of algebra, and provides carefully structured intensive practice, might reverse current trends and increase the number of students who are prepared properly for the study of calculus. It's certainly worth trying.

Endnote: Here are a few examples of expressions in which the symbols '0 + h' do not indicate adding 0 to h and in which those symbols cannot be replaced by 'h'.

a) 30 + h is not the same as 3h (of course!).

b) $2\text{^}0 + h$ is calculator notation for $2^0 + h = 1 + h,$ whereas replacing 0 + h by h would yield the incorrect expression $2\text{^}h = 2^h$.

c) 4*0 + h = h because multiplication is done before addition, whereas replacing 0 + h by h yields the incorrect expression 4*h.

Appendix

References